J. Fluid Mech. (1986), *vol.* 166, *pp.* 161–171
Printed in Great Britain

An integral equation for the floating-body problem

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(Received 20 August 1984 and in revised form 13 November 1985)

The time-harmonic three-dimensional finite-depth floating-body problem is reformulated as a boundary integral equation. Using the elementary fundamental solution that satisfies the boundary condition on the sea bottom but not the linearized free surface condition, the integral equation extends over both the ship hull and the free surface. It is shown that this integral equation is free of irregular frequencies, that is, it has at most one solution.

1. Introduction

In his classic work on the floating body problem, John (1950) showed how the boundary-value problem could be reduced to an integral equation over the wetted portion of the ship hull. The kernel of his integral operator was the Green function for the entire fluid domain with no ship present that satisfied the boundary condition at the bottom of fluid (assumed flat) and the linearized free-surface condition on the entire fluid–air boundary. John demonstrated the existence of irregular frequencies, frequencies for which the integral equation was not uniquely solvable. Recently Kleinman (1982) provided two methods of modifying the integral equation so that there were no irregular frequencies. In one case the domain of the integral operator was enlarged and in the other the operator itself changed, but both methods employed John's Green function, which is rather complicated, especially in the three-dimensional, finite-depth case.

Another way to treat this problem is to employ a much simpler Green function, one that satisfies only the boundary condition at the bottom of the fluid. Since this does not satisfy the free-surface condition, we obtain an integral equation defined over both the wetted surface of the ship hull and the free surface. Such an integral equation has been derived and even solved numerically for certain cases, e.g. Yeung (1975) and Bai & Yeung (1974). Numerical evidence indicates that this integral equation does not exhibit irregular frequencies but no conclusive analytical argument has yet appeared to support this conjecture.

The present paper provides a proof of the conjecture that this integral equation has no irregular frequencies. By irregular frequencies is meant frequencies for which the integral equation is not uniquely solvable even though the solution of the corresponding boundary-value problem is unique. What we prove is that the integral equation obtained using a simple combination of elementary sources is uniquely solvable at all frequencies.

It should be emphasized that our concern here is not with uniqueness for the boundary-value problem itself. There John required certain geometric restrictions in order to establish uniqueness. These restrictions may be somewhat relaxed to include hull forms with corners and non-normal intersections with the free surface (see Kleinman 1982). However, in the three-dimensional case treated here, we retain the restriction that vertical rays from the free surface may not intersect the ship hull in order that the boundary-value problem be uniquely solvable. Our concern here is with integral-equation formulations and the irregular frequencies that are introduced in some cases.

It should be noted that the occurrence of irregular frequencies in integral equation formulations of acoustic scattering problems is entirely analogous to the present case. (See e.g. Smirnov 1964; Brundrit 1965; Copley 1968; Schenck 1968; Chertok 1970, 1971.) However, methods for removing the irregular frequencies in acoustic scattering all essentially involve making the kernel of the integral equation more complicated (e.g. Brakhage & Werner 1965; Burton & Miller 1971; Kleinman & Roach 1974, 1982). In the present case the irregular frequencies are removed by making the kernel simpler but extending the range of integration.

2. Notation and statement of problem

Specifically, we treat the three-dimensional floating-body problem with finite depth h. If we denote the fluid domain by D_+, the hull by C_0, the free surface by C_{f} and the bottom by C_{B}, and if we denote by D_- the domain consisting of the upper-half space and the interior of the ship hull, then the geometry may be illustrated as in figure 1.

The function ϕ solves the floating-body problem if

$$\left.\begin{aligned} \nabla^2\phi = 0 \quad \text{in } D_+, \quad \frac{\partial\phi}{\partial n} = V \quad \text{on } C_0, \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } C_{\mathrm{B}}, \\ \frac{\partial\phi}{\partial n} + k\phi = 0 \quad \text{on } C_{\mathrm{f}}, \end{aligned}\right\} \tag{1}$$

and provided ϕ satisfies a radiation condition. Here $\partial/\partial n$ is the normal derivative directed into D_+ and V is a given function. The radiation condition is specified in the form

$$\frac{\partial\phi}{\partial\rho} - \mathrm{i}k_0\phi = O(\rho^{-\frac{1}{2}}) \quad \text{as } \rho\to\infty, \tag{2}$$

uniformly in θ and y. This condition may be shown to guarantee that

$$\phi(p) = \frac{\mathrm{e}^{\mathrm{i}k_0\rho}}{\rho^{\frac{1}{2}}}\,(f(\theta)+O(\rho^{-1})) \quad \text{as } \rho\to\infty, \tag{3}$$

(ρ,θ) being polar coordinates in the free-surface water plane and k_0 is the positive real root of the transcendental equation

$$k = k_0 \tanh k_0 h. \tag{4}$$

Now define the Green function

$$\gamma(p,q) = -\frac{1}{2\pi|p-q|} - \frac{1}{2\pi|p-q_1|}, \tag{5}$$

where $p = (x_p, y_p, z_p)$, $q = (x_q, y_q, z_q)$ and $q_1 = (x_q, -2h-y_q, z_q)$, and we have oriented a rectangular coordinate system so that the plane $y = 0$ is the water plane and free surface while $y = -h$ is the bottom.

With the Green function defined in (5), which has a double strength singularity on C_{B}, Green's theorem for solutions of Laplace's equation in D_+ which satisfies the radiation condition (2) takes the form

$$\int_{C_0\cup C_{\mathrm{f}}\cup C_{\mathrm{B}}} \left(\gamma(p,q)\frac{\partial\phi}{\partial n_q} - \phi(q)\frac{\partial\gamma}{\partial n_q}\right)\mathrm{d}s_q = \alpha(p)\,\phi(p), \tag{6}$$

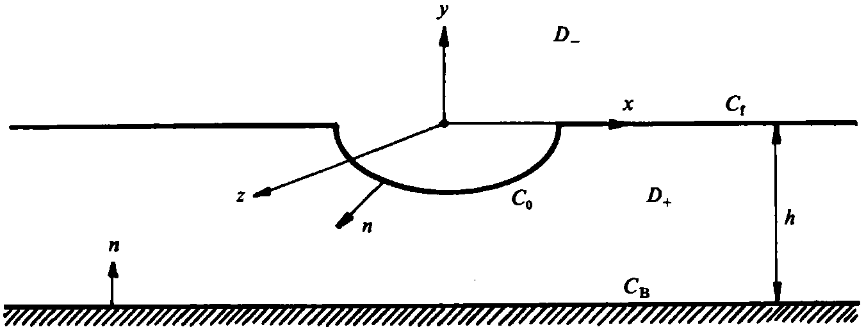

FIGURE 1

with

$$\alpha(p) := \lim_{\epsilon \to 0} \int_{[\partial B_\epsilon(p)] \cap D_+} \frac{\partial \gamma}{\partial n_q} \, ds_q, \tag{7}$$

where $\partial B_\epsilon(p)$ denotes the boundary of a ball of radius ϵ and having centre at p.

If ϕ satisfies all of the boundary conditions in (1) we obtain the boundary-integral equation

$$\alpha(p)\,\phi(p) + \int_{C_0} \phi(q) \frac{\partial \gamma}{\partial n_q}(p,q)\, ds_q + \int_{C_f} \phi(q) \left[\frac{\partial \gamma}{\partial n_q}(p,q) + k\gamma(p,q)\right] ds_q = \int_{C_0} \gamma(p,q)\, V(q)\, ds_q, \tag{8}$$

where p lies either on C_0 or C_f. The integral on C_B vanishes since both γ and ϕ satisfy a homogeneous Neumann condition and the integral over a large cylinder vanishes since $\gamma = O(\rho^{-1})$ and $\phi = O(\rho^{-\frac{1}{2}})$, the radiation condition ensuring that ϕ has asymptotic growth given by (3). As explained in the introduction, this equation has irregular frequencies if there are certain values of k for which the homogeneous equation ($V = 0$) has non-trivial solutions. We prove here that such irregular frequencies do not exist.

3. Uniqueness

Specifically our central result can be stated as follows:

THEOREM. *If (a)*

$$\phi = \frac{e^{ik_0\rho}}{\rho^{\frac{1}{2}}}(f(\theta) + O(\rho^{-1})) \quad \text{as } \rho \to \infty,$$

(b)

$$\alpha(p)\,\phi(p) + \int_{C_0} \phi(q) \frac{\partial \gamma}{\partial n_q}\, ds_q + \int_{C_f} \phi(q) \left[\frac{\partial \gamma}{\partial n_q} + k\gamma\right] ds_q = 0$$

for all $p \in C_0 \cup C_f$, and (*c*) *ϕ is continuous on* $C_0 \cup C_f$ *then* $\phi(p) \equiv 0$.

Proof. The proof of this theorem depends on the growth of potentials with densities satisfying conditions (*a*), (*b*) and (*c*) of the theorem. Assume that ϕ is a function satisfying (*a*), (*b*) and (*c*) of the theorem and define the functions u_+ and u_- in D_+ and D_- respectively as

$$\left.\begin{matrix} u_+ \\ u_- \end{matrix}\right\} := \int_{C_0} \phi(q) \frac{\partial \gamma}{\partial n_q}(p,q)\, ds_q + \int_{C_f} \phi(q) \left[\frac{\partial \gamma}{\partial n_q}(p,q) + k\gamma(p,q)\right] ds_q, \; \begin{cases} p \in D_+ \\ p \in D_- \end{cases}. \tag{9}$$

As will be seen shortly, an essential ingredient involves the growth of u_- for large radial distances from the origin. Observe that since γ has no singularities when $q \in C_0 \cup C_f$, $p \in D_-$ and γ is a solution of Laplace's equation it follows that

$$\nabla^2 u_- = 0, \quad p \in D_-. \tag{10}$$

The jump conditions for the double layer defined on $C_0 \cup C_f$ take the form

$$\lim_{\substack{p \to C_0 \cup C_f \\ p \in D_\pm}} \int_{C_0 \cup C_f} \phi(q) \frac{\partial \gamma}{\partial n_q}(p,q)\, ds_q = \left.\begin{matrix}(\alpha(p)-2) \\ \alpha(p)\end{matrix}\right\} \phi(p) + \int_{C_0 \cup C_f} \phi(q) \frac{\partial \gamma}{\partial n_q}(p,q)\, ds_q, \quad \text{for all } p \in C_0 \cup C_f. \tag{11}$$

This, together with the continuity of the single layer, implies that

$$\lim_{\substack{p \to C_0 \cup C_f \\ p \in D_-}} u_-(p) = \alpha(p)\phi(p) + \int_{C_0} \phi(q) \frac{\partial \gamma}{\partial n_q}(p,q)\, ds_q + \int_{C_f} \phi(q) \left[\frac{\partial \gamma}{\partial n_q} + k\gamma\right] ds_q, \quad \text{for all } p \in C_0 \cup C_f. \tag{12}$$

But ϕ satisfies the homogeneous equation (*b*) hence

$$\lim_{\substack{p \to C_0 \cup C_f \\ p \in D_-}} u_-(p) = 0. \tag{13}$$

However, as established in the Appendix, $\lim_{r\to\infty} u_- = 0$. Hence the maximum principle, which asserts that u_- assumes its maximum and minimum values on the boundary, implies that

$$u_- \equiv 0 \quad (p \in D_-). \tag{14}$$

Therefore

$$\frac{\partial u_-}{\partial n_-} = 0 \quad \text{on } C_0 \text{ and } C_f, \tag{15}$$

where $\partial/\partial n_-$ indicates the normal derivative from D_-. Using the defining equation (9) for u_-, we find with the usual jump conditions for the single layer

$$\frac{\partial}{\partial n_p} \int_{C_0 \cup C_f} \phi(q) \frac{\partial \gamma}{\partial n_q}(p,q)\, ds_q + k \int_{C_f} \phi(q) \frac{\partial}{\partial n_p} \gamma(p,q)\, ds_q + \beta(p)\phi(p) = 0,$$

where

$$\beta(p) = \begin{cases} 0, & p \in C_0 \\ -k, & p \in C_f. \end{cases} \tag{16}$$

Note that while existence of the normal derivative of the double layer is not guaranteed for a merely continuous density ϕ, once it is established that $u_- \equiv 0$ and hence has an ordinary normal derivative, namely zero, the defining equation for u_- ensures that the normal derivative of the double layer exists in the ordinary sense since u_- and the single layer have ordinary normal derivatives at all points where a unique normal exists.

Now examine the limiting values of u_+ as p approaches $C_0 \cup C_f$ from D_+. Using the usual jump conditions we find

$$u_+(p) = (\alpha(p)-2)\phi(p) + \int_{C_0} \phi(q) \frac{\partial \gamma}{\partial n_q}(p,q)\, ds_q + \int_{C_f} \phi(q) \left[\frac{\partial \gamma}{\partial n_q} + k\gamma\right] ds_q, \quad p \in C_0 \cup C_f \tag{17}$$

and, since ϕ satisfies the integral equation (*b*),

$$u_+(p) = -2\phi(p), \quad p \in C_0 \cup C_f. \tag{18}$$

Observe that since ϕ is assumed to have growth as specified in (*a*), equation (18) ensures that $u_+(p)$ has the same growth on C_f.

Now form the normal derivative of u_+ from D_+, obtaining

$$\frac{\partial u_+}{\partial n_+} = \frac{\partial}{\partial n_p}\int_{C_0 \cup C_f} \phi(q)\frac{\partial \gamma}{\partial n_q}(p,q)\,\mathrm{d}s_q + k\int_{C_f}\phi(q)\frac{\partial \gamma}{\partial n_p}(p,q)\,\mathrm{d}s_q - \beta(p)\,\phi(p). \tag{19}$$

Since the normal derivatives of the double layer with continuous density are the same from either side provided one of them exists, we use (16) and (18) to obtain

$$\frac{\partial u_+}{\partial n_+} = -2\beta(p)\,\phi(p) = \beta(p)\,u_+. \tag{20}$$

With the definition of $\beta(p)$ (cf. (16)) we see that

$$\frac{\partial u_+}{\partial n_+} = 0, \quad p \in C_0, \tag{21}$$

and

$$\frac{\partial u_+}{\partial n_+} = -ku_+, \quad p \in C_f. \tag{22}$$

Also,

$$\frac{\partial u_+}{\partial n_+} = 0, \quad p \in C_B \tag{23}$$

since this property is inherited from $\gamma(p,q)$. Furthermore, by its construction u_+ satisfies Laplace's equation in D_+ and since u_+ also satisfies the Neumann condition on C_B and the free-surface condition on C_f, u_+ has the representation, following John (1950),

$$u_+ = \sum_{n=0}^{\infty} u_n(\rho,\theta)\cosh k_n(y+h) \quad (\rho = (x^2+z^2)^{\frac{1}{2}} \geqslant a), \tag{24}$$

where k_n are the roots of the transcendental equation (4) and a is any number greater than the diameter of the ship hull i.e.

$$a > \max_{p \in C_0} \rho.$$

Recall that (ρ, θ, y) are the cylindrical coordinates of the point p. Moreover, as shown in the Appendix, $u_+ = O(1/\rho^{\frac{1}{2}-\delta})$, hence

$$\int_{-h}^{0} u_+(\rho,\theta,h)\cosh k_n(y+h)\,\mathrm{d}y = O(1/\rho^{\frac{1}{2}-\delta}), \tag{25}$$

which implies, with the orthogonality of $\{\cosh k_n(y+h)\}$ on $L_2(-h,0)$, that

$$u_n(\rho,\theta) = O(1/\rho^{\frac{1}{2}-\delta}). \tag{26}$$

This in turn implies that

$$\int_0^{2\pi} u_n(\rho,\theta)\,\mathrm{e}^{-\mathrm{i}m\theta}\,\mathrm{d}\theta = O(1/\rho^{\frac{1}{2}-\delta}), \tag{27}$$

and since the most general form of $u_n(\rho,\theta)$ is

$$u_n(\rho,\theta) = \sum_{m=-\infty}^{\infty}[a_{nm}H^{(1)}_{|m|}(k_n\rho) + b_{nm}H^{(2)}_{|m|}(k_n\rho)]\,\mathrm{e}^{\mathrm{i}m\theta}, \tag{28}$$

it follows that

$$a_{nm} H^{(1)}_{|m|}(k_n \rho) + b_{nm} H^{(2)}_{|m|}(k_n \rho) = O(1/\rho^{\frac{1}{2}-\delta}). \quad (29)$$

Here $H^{(1),(2)}_{|m|}$ are Hankel functions of the first and second kind respectively. The fact that k_n is positive imaginary for $n > 0$ then ensures that

$$b_{nm} = 0 \quad (n > 0).$$

Then

$$u_+(\rho, \theta, 0) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm} H^{(1)}_{|m|(k_n \rho)} \, e^{im\theta} \cosh k_n h + \sum_{m=-\infty}^{\infty} b_{0m} H^{(2)}_{|m|}(k_0 \rho) \, e^{im\theta} \cosh k_0 h, \quad (30)$$

and because $u_+(\rho, \theta, 0)$ has the same asymptotic growth as $H^{(1)}_{|m|}(k_0 \rho)$ [cf. (18)] uniformly in θ we may conclude that $b_{0m} = 0$ which then implies that

$$u_+(\rho, \theta, y) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm} H^{(1)}_{|m|}(k_n \rho) \, e^{im\theta} \cosh k_n(y+h). \quad (31)$$

Hence u_+ satisfies the radiation condition for $-h \leqslant y \leqslant 0$.

Thus u_+ is a solution of the homogeneous floating-body problem in D_+ (cf. (1) and (2)) and therefore, provided that C_0 satisfies the geometric restrictions of the uniqueness proof (John 1950; Kleinman 1982), it follows that $u_+ = 0$ in D_+ and hence also on $C_0 \cup C_f$. Equation (20) then ensures that $\phi(p) = 0$ on $C_0 \cup C_f$. That is, the only solution of the integral equation (*b*) satisfying (*a*) and (*c*) is $\phi = 0$. This means that the integral equation (7) has no irregular frequencies and has at most one solution. The existence of this solution for all k will be discussed elsewhere.

We remark that if the integral equation (8) has a solution ϕ on $C_0 \cup C_f$ then the solution of the inhomogeneous floating-body problem (1) is given by

$$\phi(p) = -\frac{1}{2} \int_{C_0} \phi(q) \frac{\partial \gamma}{\partial n_q}(p, q) \, ds_q - \frac{1}{2} \int_{C_f} \phi(q) \left[\frac{\partial \gamma}{\partial n_q} + k\gamma(p, q) \right] ds_q + \frac{1}{2} \int_{C_0} V(q) \gamma(p, q) \, ds_q \quad (32)$$

for $p \in D_+ \cup C_B$.

This research was sponsored by the Naval Sea Systems Command General Hydromechanics Research (GHR) Program administered by the David W. Taylor Naval Ship Research Contract N00014-83-K-0060. The authors also wish to acknowledge many helpful comments by the referees. Particular thanks are due Professor R. Kress of Göttingen for his careful analysis and critical observations of an earlier draft which occasioned revisions embodied in the present paper.

Appendix. On the growth of $u_\pm$

Here we prove the Lemma needed in establishing uniqueness of solutions of the integral equation. For convenience we restate it as follows:

LEMMA. *If* (*a*)

$$\phi = \frac{e^{ik_0 \rho}}{\rho^{\frac{1}{2}}} (f(\theta) + O(\rho^{-1})) \quad \text{as } \rho \to \infty,$$

(*b*) $\alpha(p)\,\phi(p)+\int_{C_0}\phi(q)\,\dfrac{\partial\gamma}{\partial n_q}\,\mathrm{d}s_q+\int_{C_f}\phi(q)\left[\dfrac{\partial\gamma}{\partial n_q}+k\gamma\right]\mathrm{d}s_q=0,\quad p\in C_0\cup C_f,$

(*c*) *ϕ is continuous on $C_0\cup C_f$,*

and

(*d*) $u_{\pm}=\int_{C_0}\phi(q)\,\dfrac{\partial\gamma}{\partial n_q}\,\mathrm{d}s_q+\int_{C_f}\phi(q)\left[\dfrac{\partial\gamma}{\partial n_q}+k\gamma\right]\mathrm{d}s_q\quad(p\in D_{\pm}),$

then

$$u_-=O\left(\frac{1}{r_p^{\frac{1}{2}-\delta}}\right),\quad \textit{as } r_p\to\infty,\quad \delta<\tfrac{1}{2}$$

and

$$u_+=O\left(\frac{1}{\rho_p^{\frac{1}{2}-\delta}}\right)\quad \textit{as } \rho_p\to\infty,$$

where $r_p=|p|=(\rho_p^2+y_p^2)^{\frac{1}{2}}$.

Proof. With γ as defined in (5) it is clear that

$$\int_{C_0}\phi(q)\,\frac{\partial\gamma}{\partial n_q}\,\mathrm{d}s_q=O\left(\frac{1}{r_p^2}\right), \tag{A 1}$$

and

$$\int_{C_f\cup B_a}\phi(q)\left[\frac{\partial\gamma}{\partial n_q}+k\gamma\right]\mathrm{d}s_q=O\left(\frac{1}{r_p}\right), \tag{A 2}$$

where $C_f\cap B_a$ is that portion of the free surface contained in the ball of radius a. It is a bit more work to establish the growth of

$$\int_{C_f\cap B_a^c}\phi(q)\left[\frac{\partial\gamma}{\partial n_q}+k\gamma\right]\mathrm{d}s_q,$$

where B_a^c is the complement of the ball. Considering first the term involving the normal derivative, which on C_f is

$$\frac{\partial}{\partial n_q}=-\frac{\partial}{\partial y_q}\bigg|_{y_q=0},$$

we find that

$$\int_{C_f\cap B_a^c}\phi(q)\,\frac{\partial\gamma}{\partial n_q}\,\mathrm{d}s_q=\frac{1}{2\pi}\int_0^{2\pi}\int_a^{\infty}\phi(q)\left(\frac{y_p}{R(0)^3}-\frac{y_p+2h}{R^3(h)}\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \tag{A 3}$$

where (ρ,θ) are the cylindrical coordinates of q on C_f and

$$R(h)=((x_p-x_q)^2+(z_p-z_q)^2+(y_p+2h)^2)^{\frac{1}{2}}.$$

Introduce two sets of spherical coordinates of the form

$$\begin{aligned} z_p&=r_p\sin\alpha\cos\theta_p & &\qquad & z_p&=r'\sin\alpha'\cos\theta_p\\ x_p&=r_p\sin\alpha\sin\theta_p & &\text{and} & x_p&=r'\sin\alpha'\sin\theta_p\\ y_p&=r_p\cos\alpha & & & y_p+2h&=r'\cos\alpha'\end{aligned}$$

where $0\leqslant\theta_p\leqslant 2\pi$, $0\leqslant\alpha\leqslant\pi$, $0\leqslant\alpha'\leqslant\frac{1}{2}\pi$, $r_p, r'\geqslant 0$. Clearly (r_p,θ_p,α) are the usual spherical coordinates while r' and α' will depend on h. Explicitly,

$$r'=(x_p^2+z_p^2+(y_p+2h)^2)^{\frac{1}{2}}=(r_p^2+2hy_p+4h^2)^{\frac{1}{2}},\quad y'=y_p+2h,$$

hence $r_p/r'\leqslant 1$ for $2h^2+hy_p\geqslant 0$, a condition always satisfied for $p\in\overline{D_+\cup D_-}$. Note that $y_p \geqslant -h$ when $p \in \overline{D_+ \cup D_-}$ hence $\alpha' \leqslant \frac{1}{2}\pi$ whereas α varies over a larger interval, in fact, $\alpha > \frac{1}{2}\pi$ when $p \in D_+$. In this notation

$$R(h) = (r'^2 + \rho^2 - 2r'\rho \sin\alpha' \cos(\theta - \theta_p))^{\frac{1}{2}}, \tag{A 4}$$

and

$$\int_{C_f \cap B_a^c} \phi(q) \frac{\partial \gamma}{\partial n_q} \mathrm{d}s_q = \frac{1}{2\pi} \int_0^{2\pi} \int_a^\infty \phi(q) \left(\frac{r_p \cos\alpha}{(r_p^2 + \rho^2 - 2r\rho\, (\sin\alpha \cos(\theta - \theta_p)^{\frac{3}{2}}))} - \frac{r' \cos\alpha'}{r'^2 + \rho^2 - 2r'\rho \sin\alpha' \cos(\theta - \theta_p)^{\frac{3}{2}}} \right) \rho \,\mathrm{d}\rho \,\mathrm{d}\theta. \tag{A 5}$$

It suffices to consider the first integral on the right, the analysis for the second being identical, with r', α', y' replacing r_p, α, y_p. For brevity we omit the subscript and denote r_p by r in the ensuing analysis and consider

$$r \cos\alpha \int_0^{2\pi} \int_a^\infty \frac{\phi(q)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\theta}{(r^2 + \rho^2 - 2r\rho \sin\alpha \cos(\theta - \theta_p))^{\frac{3}{2}}} \quad \text{for } 0 \leqslant \alpha \leqslant \pi. \tag{A 6}$$

Using the asymptotic form of ϕ and the substitution $\rho = rt$ we find

$$r \cos\alpha \int_0^{2\pi} \int_a^\infty \frac{\phi(q)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\theta}{(r^2 + \rho^2 - 2r\rho \sin\alpha \cos(\theta - \theta_p))^{\frac{3}{2}}} = \frac{\cos\alpha}{r^{\frac{1}{2}}} \int_0^{2\pi} \int_{a/r}^\infty \frac{\mathrm{e}^{ik_0 rt}}{t^{\frac{1}{2}}} \frac{(f(\theta) + O(1))\, t \,\mathrm{d}t \,\mathrm{d}\theta}{(1 + t^2 - 2t \sin\alpha \cos(\theta - \theta_p))^{\frac{3}{2}}}.$$

Hence the integral is $O(r^{-\frac{1}{2}})$ for $\alpha \neq \frac{1}{2}\pi$. Note that this expression does not obviously exist when $\alpha \to \frac{1}{2}\pi$. To see what happens as $\alpha \to \frac{1}{2}\pi$ observe that

$$\begin{aligned} &\lim_{\alpha \to \frac{1}{2}\pi \pm} r \cos\alpha \int_0^{2\pi} \int_a^\infty \frac{\phi(q)\, \rho \,\mathrm{d}\rho \,\mathrm{d}\theta}{(r^2 + \rho^2 - 2r\rho \sin\alpha \cos(\theta - \theta_p))^{\frac{3}{2}}} \\ &= \lim_{y_p \to 0 \mp} - \int_0^{2\pi} \int_a^\infty \phi(q) \frac{\mathrm{d}}{\mathrm{d}y_q} \frac{1}{((x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2)^{\frac{1}{2}}} \bigg|_{y_q = 0} \rho \,\mathrm{d}\rho \,\mathrm{d}\theta \\ &= \lim_{\substack{p \to C_f \\ p \in D_\pm}} \pm 2\pi \int_{C_f \cup B_a^c} \phi(q) \frac{\partial}{\partial n_q} \gamma_0(p, q) \,\mathrm{d}s_q, \\ &= \pm 2\pi \phi(p), \quad \rho_p > a, \end{aligned}$$

where

$$\gamma_0 = -\frac{1}{2\pi((x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2)^{\frac{1}{2}}},$$

and the jump condition for a double layer is used. Here we make no use of the assumption that ϕ is a solution of the integral equation (*b*). The integral in the jump condition vanishes for p on C_f, $(y_p = 0)$. Now we use (*a*) which asserts that on C_f, ϕ is assumed to grow as $O(\rho^{-\frac{1}{2}})$ which is the desired growth. Hence the integral (A 6) is $O(r^{-\frac{1}{2}})$ for $0 \leqslant \alpha \leqslant \pi$. Redoing the analysis with r', α', y' replacing r_p, α, y_p leads to a similar result. Hence we conclude that

$$\int_{C_f \cap B_a^c} \phi(q) \frac{\partial \gamma}{\partial n_q}(p, q) \,\mathrm{d}s_q = O(r^{-\frac{1}{2}}) \quad \text{as } r \to \infty, \quad y_p \geqslant -h. \tag{A 7}$$

Next we consider

$$\int_{C_f \cap B_a^c} \phi(q)\, \gamma(p, q) \,\mathrm{d}s_q = -\frac{1}{2\pi} \int_0^{2\pi} \int_a^\infty \phi(q) \left[\frac{1}{R(0)} + \frac{1}{R(h)} \right] \rho \,\mathrm{d}\rho \,\mathrm{d}\theta. \tag{A 8}$$

Using the notation previously introduced and the asymptotic form of ϕ we must treat integrals of the form

$$I = \int_0^{2\pi}\int_a^\infty \frac{e^{ik_0\rho}}{\rho^{\frac{1}{2}}}\frac{[f(\theta)+O(1/\rho)]\,\rho\,d\rho\,d\theta}{(r^2+\rho^2-2r\rho\sin\alpha\cos(\theta-\theta_p))^{\frac{1}{2}}}, \tag{A 9}$$

and a similar integral with r', α' replacing r, α.

The term involving $O(1/\rho)$ is easily handled since

$$\begin{aligned}\left|\int_0^{2\pi}\int_a^\infty \frac{e^{ik_0\rho}}{\rho^{\frac{1}{2}}}\frac{O(1/\rho)\,\rho\,d\rho\,d\theta}{(r^2+\rho^2-2r\rho\sin\alpha\cos(\theta-\theta_p))^{\frac{1}{2}}}\right| &\leqslant c\int_0^{2\pi}\int_a^\infty \frac{d\rho\,d\theta}{\rho^{\frac{1}{2}}(r^2+\rho^2-2r\rho\sin\alpha\cos\theta)^{\frac{1}{2}}}\\ &\leqslant \frac{c}{r^{\frac{1}{2}}}\int_0^{2\pi}\int_0^\infty \frac{dt\,d\theta}{t^{\frac{1}{2}}(1+t^2-2t\cos\theta)^{\frac{1}{2}}}\end{aligned} \tag{A 10}$$

where c is independent of r and α. This is seen to be $O(r^{-\frac{1}{2}})$ since the integral on the right exists and is independent of r.

The remaining integral is of the form

$$\begin{aligned}I_1 &= \int_0^{2\pi}\int_a^\infty \frac{e^{ik_0\rho}\,\rho^{\frac{1}{2}}f(\theta)\,d\rho\,d\theta}{(r^2+\rho^2-2r\rho\sin\alpha\cos(\theta-\theta_p))^{\frac{1}{2}}}\\ &= \frac{-1}{ik_0}\int_0^{2\pi}\frac{f(\theta)\,a^{\frac{1}{2}}\,e^{ik_0a}\,d\theta}{(r^2+a^2-2ar\sin\alpha\cos(\theta-\theta_p))^{\frac{1}{2}}}\\ &\quad -\frac{1}{ik_0}\int_0^{2\pi}\int_a^\infty e^{ik_0\rho}\frac{d}{d\rho}\left(\frac{\rho^{\frac{1}{2}}}{(r^2+\rho^2-2r\rho\sin\alpha\cos(\theta-\theta_p))^{\frac{1}{2}}}\right)d\rho\,d\theta.\end{aligned} \tag{A 11}$$

The first term on the right is clearly $O(1/r)$. Hence, on performing the indicated differentiation, we have

$$I_1 = O\left(\frac{1}{r}\right)-\frac{1}{2ik_0}\int_0^{2\pi}\int_a^\infty \frac{e^{ik_0\rho}(r^2-\rho^2)f(\theta)\,d\rho\,d\theta}{\rho^{\frac{1}{2}}(r^2+\rho^2-2r\rho\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}}$$

and letting $\rho = rt$

$$I_1 = O\left(\frac{1}{r}\right)-\frac{1}{2ik_0r^{\frac{1}{2}}}\int_0^{2\pi}\int_{a/r}^\infty \frac{e^{ik_0rt}(1-t^2)f(\theta)}{t^{\frac{1}{2}}(1+t^2-2t\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}}\,dt\,d\theta.$$

We break up the t integration into three parts and use the estimates

$$\left|\int_0^{2\pi}\int_{a/r}^{\frac{1}{2}} \frac{e^{ik_0rt}(1-t^2)f(\theta)\,dt\,d\theta}{t^{\frac{1}{2}}(1+t^2-2t\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}}\right| \leqslant c\|f\|_\infty\int_0^{\frac{1}{2}}\frac{(1-t^2)\,dt}{t^{\frac{1}{2}}(t-1)^3} = c_1\|f\|_\infty,$$

and

$$\left|\int_0^{2\pi}\int_2^\infty \frac{e^{ik_0rt}(1-t^2)f(\theta)\,dt\,d\theta}{t^{\frac{1}{2}}(1+t^2-2t\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}}\right| \leqslant c\|f\|_\infty\int_2^{\frac{1}{2}}\frac{(t^2-1)\,dt}{2^{\frac{1}{2}}(t-1)^3} = c_2\|f\|_\infty,$$

where $\|\cdot\|_\infty$ is the sup norm and the constants c_1 and c_2 are independent of α, θ_p, r and f, to obtain

$$I_1 = O(r^{-\frac{1}{2}})-\frac{1}{2ik_0r^{\frac{1}{2}}}\int_0^{2\pi}\int_{\frac{1}{2}}^2 \frac{e^{ik_0rt}(1-t^2)f(\theta)\,dt\,d\theta}{t^{\frac{1}{2}}(1+t^2-2t\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}},$$

which we write as

$$I_1 = O(r^{-\frac{1}{2}}) - \frac{1}{2ik_0 r^{\frac{1}{2}}} \int_0^{2\pi} f(\theta) \left\{ \int_{\frac{1}{2}}^{1} \frac{e^{ik_0 rt}(1-t^2)\,dt}{t^{\frac{1}{2}}(1+t^2-2t\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}} + \int_1^2 \frac{e^{ik_0 ru}(1-u^2)\,du}{u^{\frac{1}{2}}(1+u^2-2u\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}} \right\} d\theta.$$

Letting $u = 1/t$ in the second integral we get

$$I_1 = O\left(\frac{1}{r^{\frac{1}{2}}}\right) - \frac{1}{2ik_0 r^{\frac{1}{2}}} \int_0^{2\pi} f(\theta) \int_{\frac{1}{2}}^{1} \frac{(e^{ik_0 rt} - e^{ik_0 r/t})(1-t^2)\,dt}{t^{\frac{1}{2}}(1+t^2-2t\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}}. \tag{A 12}$$

The integral in (A 12) which we denote as I_2 satisfies the inequality

$$I_2 = \left| \int_0^{2\pi} f(\theta) \int_{\frac{1}{2}}^{1} \frac{(e^{ik_0 rt} - e^{ik_0 r/t})(1-t^2)\,dt}{t^{\frac{1}{2}}(1+t^2-2t\sin\alpha\cos(\theta-\theta_p))^{\frac{3}{2}}} \right|$$
$$\leqslant \|f\|_\infty \int_0^{2\pi} \int_{\frac{1}{2}}^{1} \frac{|e^{ik_0 rt} - e^{ik_0 r/t}|^{\delta}\,|e^{ik_0 rt} - e^{ik_0 r/t}|^{1-\delta}}{t^{\frac{1}{2}}(1+t^2-2t\sin\alpha\cos\theta)^{\frac{3}{2}}} (1-t^2)\,dt\,d\theta \tag{A 13}$$

for arbitrary $\delta \in (0,1)$ (we further restrict δ subsequently) and using the estimates

$$\frac{1+t}{t^{\frac{1}{2}}} \leqslant 2\,2^{\frac{1}{2}} \quad (\tfrac{1}{2} \leqslant t \leqslant 1),$$
$$|e^{ik_0 rt} - e^{ik_0 r/t}| \leqslant 2,$$
$$|e^{ik_0 rt} - e^{ik_0 r/t}| \leqslant 4k_0 r(1-t) \quad (\tfrac{1}{2} \leqslant t \leqslant 1),$$

we obtain

$$I_2 \leqslant c\|f\|_\infty r^{\delta} \int_0^{2\pi} \int_{\frac{1}{2}}^{1} \frac{(1-t)^{1+\delta}\,dt\,d\theta}{(1+t^2-2t\sin\alpha\cos\theta))^{\frac{3}{2}}}, \tag{A 14}$$

were c is independent of r, α, θ_p and f.

But for $0 \leqslant \alpha \leqslant \pi$ and $0 \leqslant \theta \leqslant 2\pi$ we may show that

$$\frac{1}{1+t^2-2t\sin\alpha\cos\theta} \leqslant \frac{2}{1+t^2-2t\cos\theta} \leqslant \frac{2}{(1-t)^2}.$$

hence

$$I_2 \leqslant c_1\|f\|_\infty r^{\delta} \int_0^{2\pi} \int_{\frac{1}{2}}^{1} \frac{dt\,d\theta}{(1+t^2-2t\cos\theta)^{1-\frac{1}{2}\delta}}.$$

The kernel is weakly singular at $t = 1$, $\theta = 0$ and hence the integral exists. Thus there is a constant, c_2, such that

$$I_2 \leqslant c_2\|f\|_\infty r^{\delta},$$

which with (A 12) establishes that

$$I_1 = O(r^{\delta-\frac{1}{2}}). \tag{A 15}$$

We may choose $\delta \in (0, \frac{1}{2})$ to ensure that I_1 decays with r. A similar growth estimate is obtained if r', α' replace r, α. Hence, with (A 8) we see that

$$\int_{C_f \cap B_a^c} \phi(q)\,\gamma(p,q)\,ds_q = O(r^{\delta-\frac{1}{2}}). \tag{A 16}$$

This result, taken together with (A 7), ensures that

$$\int_{C_f \cap B_a^c} \phi(q) \left[\frac{\partial \gamma}{\partial n_q} + k\gamma\right] ds_q = O(r^{\delta - \frac{1}{2}}), \tag{A 17}$$

which with (A 1) and (A 2) establishes that

$$u_{\pm} = O(r_p^{\delta - \frac{1}{2}}), \tag{A 18}$$

which implies, for $-h \leqslant u \leqslant 0$, that

$$u_{+} = O(\rho_p^{\delta - \frac{1}{2}}).$$

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